





A MAGIC FORMULA OF NATURE

Riccione 2018

Primo Brandi - Anna Salvadori

The description of the forms is one of the major problems of biology.

Is mathematics able to give a support?

Mathematics is the language of Science and Technology

Johan Gielis (American Journal of Botany 2003) proposed a formula that can describe a wide range of natural shapes

$$\rho = R(\varphi)\left(\left|\frac{1}{a}\cos\left(\frac{m}{4}\varphi\right)\right|^{p_2} + \left|\frac{1}{b}\sin\left(\frac{m}{4}\varphi\right)\right|^{p_3}\right)^{-\frac{1}{p_1}}$$

Gielis superformula

Bottom up: to discover the the main idea behind

Top down: to understand the role of each parameter

$$\rho = R(\varphi)\left(\left|\frac{1}{a}\cos\left(\frac{m}{4}\varphi\right)\right|^{p_2} + \left|\frac{1}{b}\sin\left(\frac{m}{4}\varphi\right)\right|^{p_3}\right)^{-\frac{1}{p_1}}$$

Gielis superformula

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Product of two functions

$$\rho = R(\varphi)\left(\left|\frac{1}{a}\cos\left(\frac{m}{4}\varphi\right)\right|^{p_2} + \left|\frac{1}{b}\sin\left(\frac{m}{4}\varphi\right)\right|^{p_3}\right)^{-\frac{1}{p_1}}$$

Gielis superformula

Bottom up: to discover the the main idea behind

Let us concentrate our attempition on the second function assuming constant the first one

$$\rho = \left(\left| \frac{1}{a} \cos\left(\frac{m}{4} \varphi \right) \right|^{p_2} + \left| \frac{1}{b} \sin\left(\frac{m}{4} \varphi \right) \right|^{p_3} \right)^{-\frac{1}{p_1}}$$

Gielis superformula

Bottom up: to discover the the main idea behind

Assume the three power parameters coincide

$$\rho = \left(\left| \frac{1}{a} \cos\left(\frac{m}{4} \varphi \right) \right|^{p_2} + \left| \frac{1}{b} \sin\left(\frac{m}{4} \varphi \right) \right|^{p_3} \right)^{-\frac{1}{p_1}}$$

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$$\rho = \left(\left| \frac{1}{a} \cos\left(\frac{m}{4} \varphi \right) \right|^{p} + \left| \frac{1}{b} \sin\left(\frac{m}{4} \varphi \right) \right|^{p} \right)^{-\frac{1}{p}}$$

Gielis superformula

Bottom up: to discover the the main idea behind

Equivalent formulation

$$\rho = \left(\left| \frac{1}{a} \cos\left(\frac{m}{4} \varphi \right) \right|^p + \left| \frac{1}{b} \sin\left(\frac{m}{4} \varphi \right) \right|^p \right)^{-\frac{1}{p}}$$

$$\rho^{-p} = \left| \frac{1}{a} \cos\left(\frac{m}{4} \varphi \right) \right|^p + \left| \frac{1}{b} \sin\left(\frac{m}{4} \varphi \right) \right|^p$$

$$1 = \left| \frac{1}{a} \rho \cos\left(\frac{m}{4} \varphi \right) \right|^p + \left| \frac{1}{b} \rho \sin\left(\frac{m}{4} \varphi \right) \right|^p$$

Gielis superformula

Bottom up: to discover the the main idea behind

$$1 = \left|\frac{1}{a}\rho\cos\left(\frac{m}{4}\varphi\right)\right|^p + \left|\frac{1}{b}\rho\sin\left(\frac{m}{4}\varphi\right)\right|^p$$

Gielis superformula

Bottom up: to discover the the main idea behind

Re-scale the variable

$$1 = \left|\frac{1}{a}\rho\cos\left(\frac{m}{4}\varphi\right)\right|^{p} + \left|\frac{1}{b}\rho\sin\left(\frac{m}{4}\varphi\right)\right|^{p}$$

Gielis superformula

Bottom up: to discover the the main idea behind

Re-scale the variable **m=4**

$$1 = \left|\frac{1}{a}\rho\cos\varphi\right|^{p} + \left|\frac{1}{b}\rho\sin\varphi\right|^{p}$$

Gielis superformula

Bottom up: to discover the the main idea behind

$$x = \rho \sin \varphi$$
$$y = \rho \cos \varphi$$

From polar to cartesian coordinates

$$1 = \left| \frac{1}{a} \rho \cos \varphi \right|^p + \left| \frac{1}{b} \rho \sin \varphi \right|^p$$

Gielis superformula

Bottom up: to discover the the main idea behind

$$x = \rho \sin \varphi$$

$$y = \rho \cos \varphi$$

From polar to cartesian coordinates

$$1 = \left|\frac{1}{a} x\right|^p + \left|\frac{1}{b} y\right|^p$$

Gielis superformula

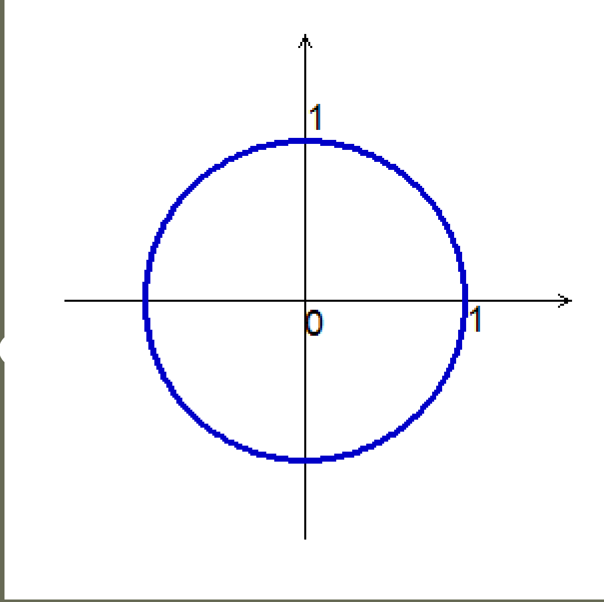
Bottom up: to discover the the main idea behind

Re-scale the two viariablex **a=b=1**

$$1 = \left|\frac{1}{a}x\right|^p + \left|\frac{1}{b}y\right|^p$$

Gielis superformula

Bottom up: to discover the the main ide



Well known equation

$$1 = |x|^p + |y|^p$$

$$1 = x^2 + y^2$$

key idea

Botton up: to discover the the main idea behind

Top down: to understand the role of parameters

$$\rho = R(\varphi)\left(\left|\frac{1}{a}\cos\left(\frac{m}{4}\varphi\right)\right|^{p_2} + \left|\frac{1}{b}\sin\left(\frac{m}{4}\varphi\right)\right|^{p_3}\right)^{-\frac{1}{p_1}}$$

Gielis superformula

Bottom up: to discover the the main idea behind

Let us start from the key idea

$$1 = |x|^p + |y|^p$$

$$1 = x^2 + y^2$$

The *squared* circle
Lamé circumference

$$r^p = |x|^p + |y|^p$$

Gabriel Lamé (1795 –1870)
revolutionized this view

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Lamé circumference

$$r^p = |x|^p + |y|^p$$

For a long time the circle and the square have been considered as "opposed" figures.

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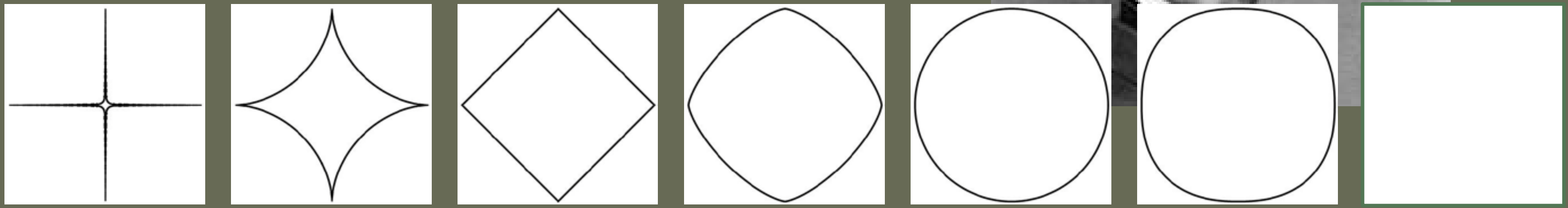

The *squared* circle
Lamé circumference

$$r^p = |x|^p + |y|^p$$

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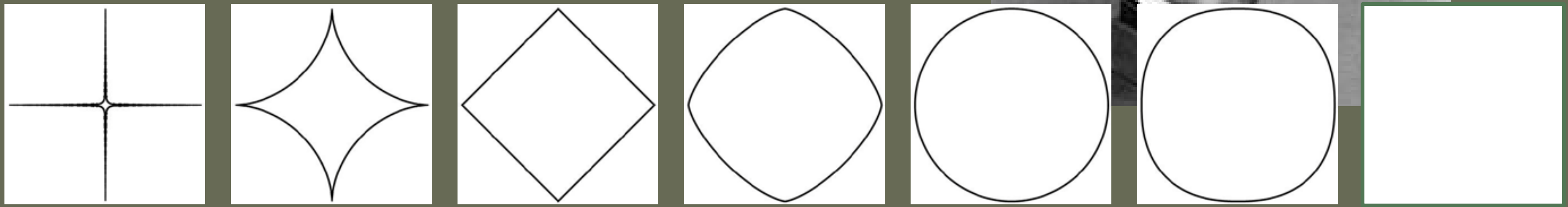
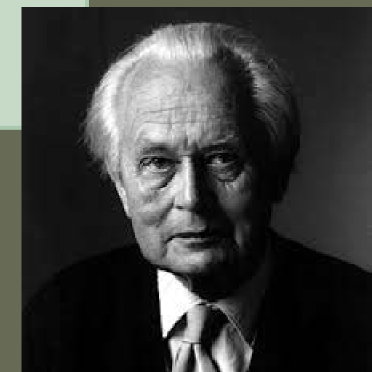
Gabriel Lamé (1795 –1870) revolutionized this view



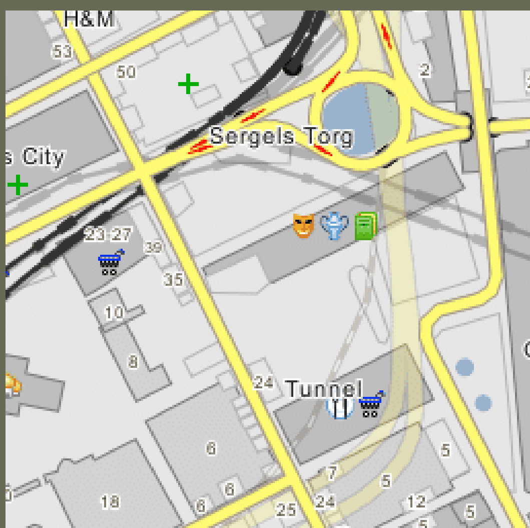
p=0	0<p<1	L_1 **Manhattan** p=1	1<p<2	**Euclidean** p=2	p>2	$L_{infinuty}$ **MAX** p->infinity

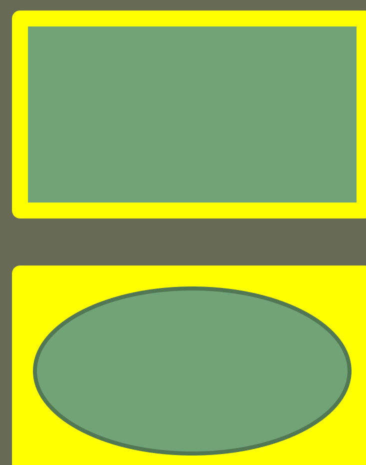
Super ellipses
In the real life

$$r^p = \left|\frac{x}{a}\right|^p + \left|\frac{y}{b}\right|^p$$



Piet Hein (1959) Sergel's Torg, Stockholm





Super ellipses
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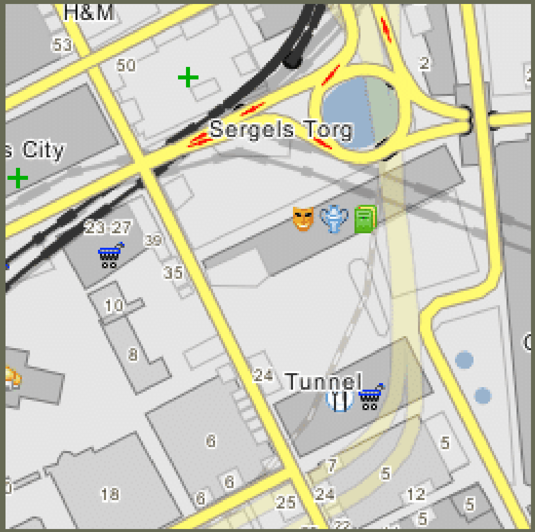
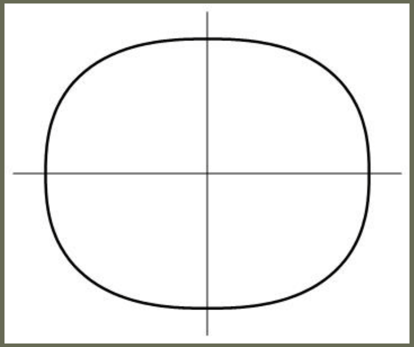
$$r^p = \left|\frac{x}{a}\right|^p + \left|\frac{y}{b}\right|^p$$

Piet Hein (1959) Sergel's Torg, Stockholm



$p = 5/2$



$a/b = 6/5$

Super ellipses
In the real life

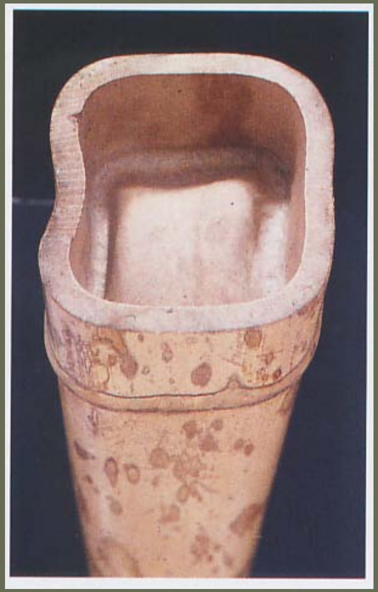
$$r^p = \left|\frac{x}{a}\right|^p + \left|\frac{y}{b}\right|^p$$

Piet Hein (1905-1996) glasses, plates, desk lamps …

Super ellipses
In the real life

$$r^p = \left|\frac{x}{a}\right|^p + \left|\frac{y}{b}\right|^p$$

bamboo cane



First step to superformula

From Cartesian to Polar coordinates

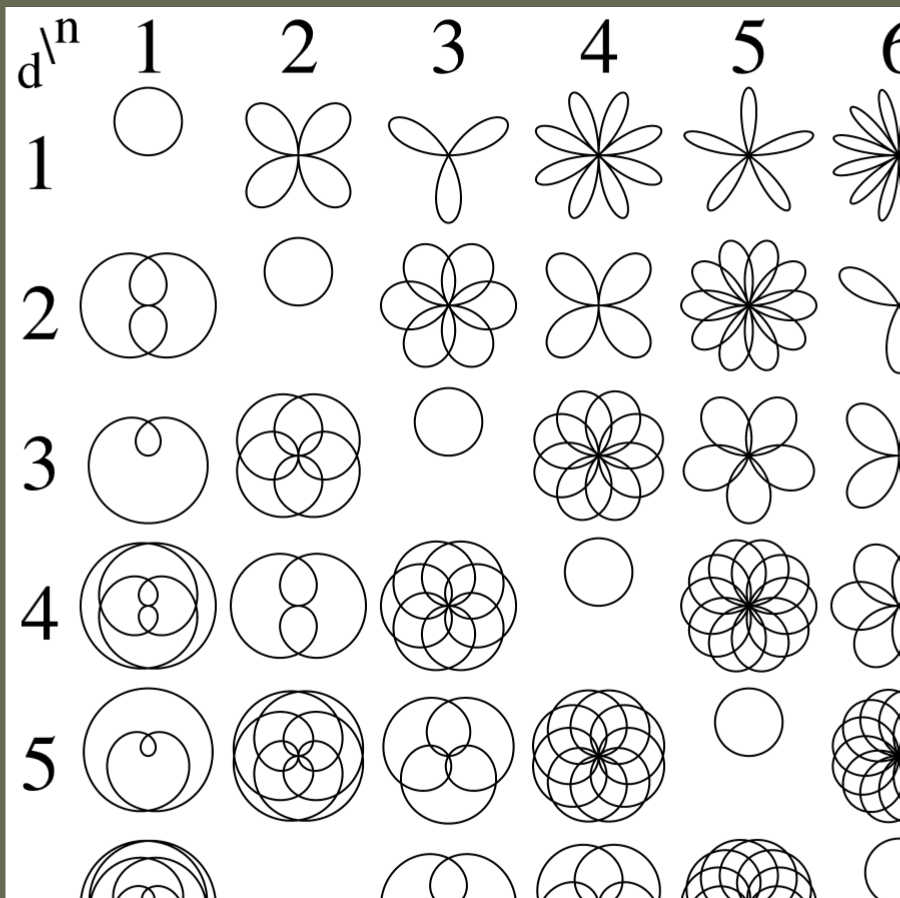
$$x = \rho \sin \varphi$$

$$y = \rho \cos \varphi$$

$$1 = \left|\frac{x}{a}\right|^p + \left|\frac{y}{b}\right|^p \quad \Leftrightarrow \quad 1 = \left|\frac{1}{a}\rho \sin \varphi\right|^p + \left|\frac{1}{b}\rho \cos \varphi\right|^p$$

First step to superformula

From Cartesian to Polar coordinates



Rodonee

Grandi's roses

Luigi Guido Grandi (1671-1742)

$$\rho = R \sin(\omega\varphi)$$

First step to superformula

From Cartesian to Polar coordinates

$$x = \rho \sin \varphi$$

$$y = \rho \cos \varphi$$

$$1 = \left|\frac{x}{a}\right|^{p} + \left|\frac{y}{b}\right|^{p} \quad \Leftrightarrow \quad 1 = \left|\frac{1}{a}\rho \sin \varphi\right|^{p} + \left|\frac{1}{b}\rho \cos \varphi\right|^{p}$$

$$\rho^{-p} = \left|\frac{1}{a}\sin \varphi\right|^{p} + \left|\frac{1}{b}\cos \varphi\right|^{p}$$

First step to superformula
From Cartesian to Polar coordinates

$$x = \rho \sin \varphi$$

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$$\rho = \left(\left|\frac{1}{a}\sin \varphi\right|^p + \left|\frac{1}{b}\cos \varphi\right|^p\right)^{-1/p}$$

First step to superformula
From Cartesian to Polar coordinates

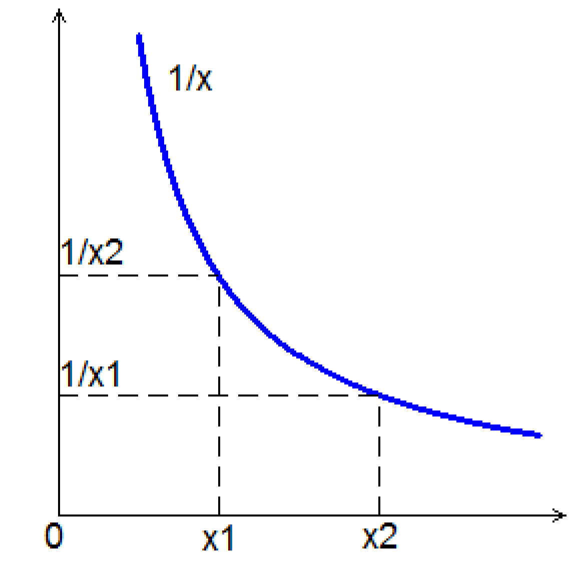
$$\rho = \left(\left| \frac{1}{a} \cos\varphi \right|^p + \left| \frac{1}{b} \sin\varphi \right|^p \right)^{-\frac{1}{p}}$$

ρ represent the **length of the vector ray** corresponding to angle φ

the local minima and maxima play a fundamental for the figure shape

First step to superformula
From Cartesian to Polar coordina

$$\rho = \left(\left| \frac{1}{a} \cos\varphi \right|^p + \left| \frac{1}{b} \sin\varphi \right|^p \right)^{-\frac{1}{p}}$$



ρ represent the **length of the vector ray** corresponding to angle φ

the local minima and maxima play a fundamental for the figure shape

They correspond to the **minimum and maximum points** of the **reciprocal function**

$$\rho(\varphi_0) \le \rho(\varphi) \quad \forall \varphi \in I \quad \Leftrightarrow \quad 1/\rho(\varphi_0) \ge 1/\rho(\varphi) \quad \forall \varphi \in I$$

First step to superformula
From Cartesian to Polar coordinates

$$\rho = \left(\left| \frac{1}{a} \cos \varphi \right|^p + \left| \frac{1}{b} \sin \varphi \right|^p \right)^{-\frac{1}{p}}$$

ρ represent the **length of the vector ray** corresponding to angle φ

the local minima and maxima play a fundamental for the figure shape

They correspond to the **minimum and maximum points** of the **reciprocal function**

$$f = 1/\rho = \left(|\cos \varphi|^p + |\sin \varphi|^p \right)^{\frac{1}{p}}$$

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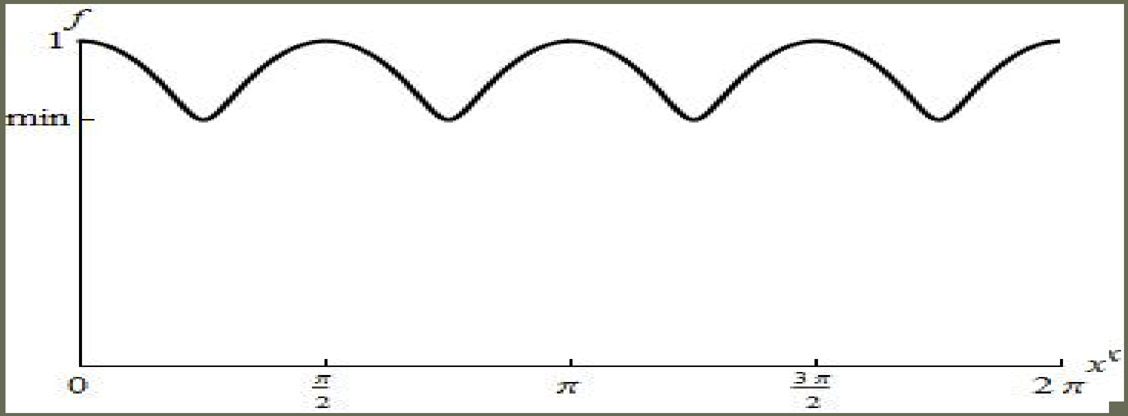
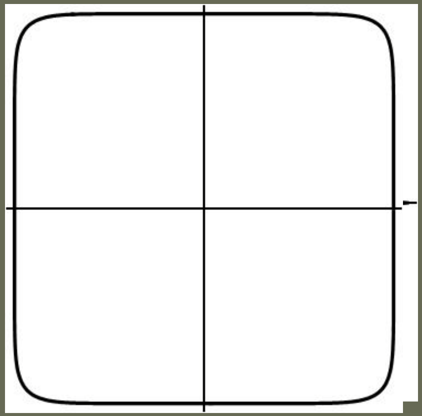
First step to superformula
From Cartesian to Polar coordinates

$$\rho = \left(\left| \frac{1}{a} \cos\varphi \right|^p + \left| \frac{1}{b} \sin\varphi \right|^p \right)^{-\frac{1}{p}}$$

f 

 ρ

Functions f admits **4 minimum points** and **4 maximum points** for every value of parameter *p*

Second step to superformula

Fase parameter

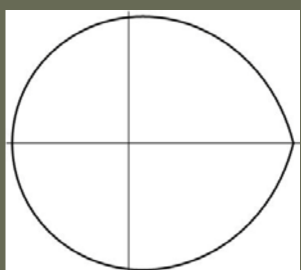
$$\rho = \left(\left| \frac{1}{a} \cos\left(\frac{m}{4} \varphi \right) \right|^{p} + \left| \frac{1}{b} \sin\left(\frac{m}{4} \varphi \right) \right|^{p} \right)^{-\frac{1}{p}}$$

Second step to superformula

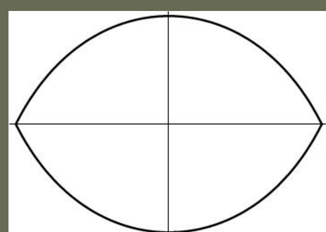
Fase parameter

$$\rho = \left(\left| \frac{1}{a} \cos\left(\frac{m}{4} \varphi \right) \right|^{p} + \left| \frac{1}{b} \sin\left(\frac{m}{4} \varphi \right) \right|^{p} \right)^{-\frac{1}{p}}$$

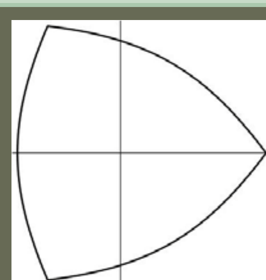
m integer



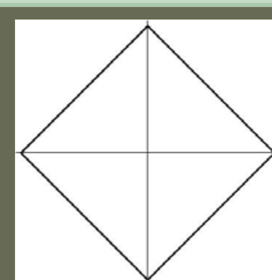
m=1



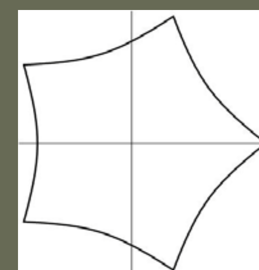
m=2



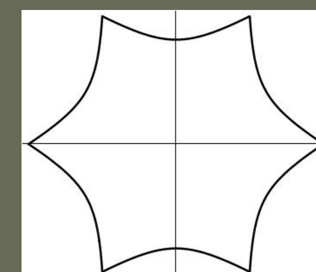
m=3



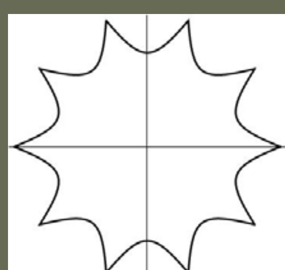
m=4



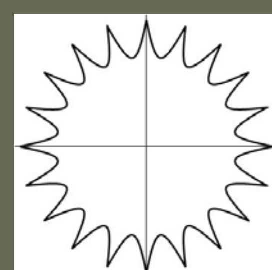
m=5



m=6



m=10

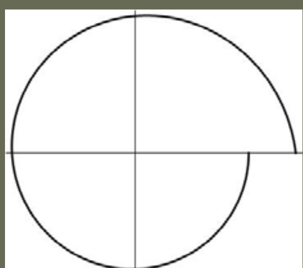


m=20

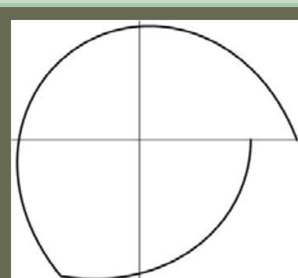
Second step to superformula
Fase parameter

$$\rho = \left(\left| \frac{1}{a} \cos\left(\frac{m}{4} \varphi \right) \right|^{p} + \left| \frac{1}{b} \sin\left(\frac{m}{4} \varphi \right) \right|^{p} \right)^{-\frac{1}{p}}$$

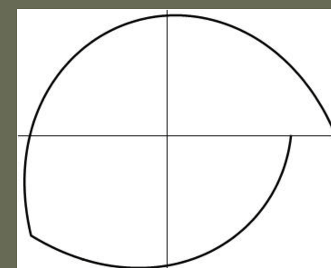
m rational



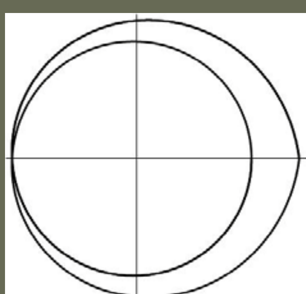
m=1/2



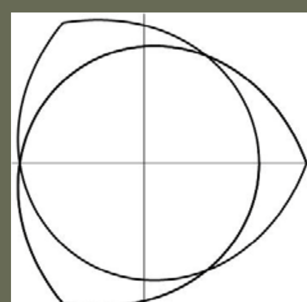
m=3/2



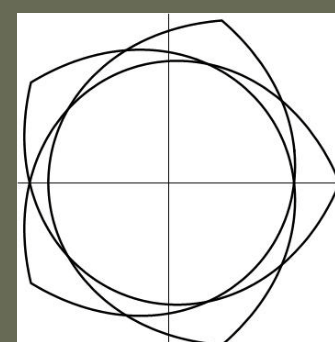
m=5/3



2 spins



2 spins



3 spins

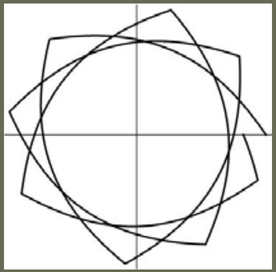
Second step to superformula

Fase parameter

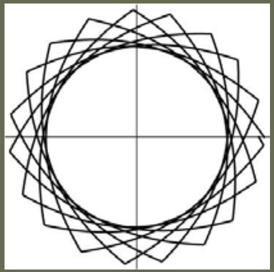
$$\rho = \left(\left| \frac{1}{a} \cos\left(\frac{m}{4} \varphi \right) \right|^{p} + \left| \frac{1}{b} \sin\left(\frac{m}{4} \varphi \right) \right|^{p} \right)^{-\frac{1}{p}}$$

m irrational

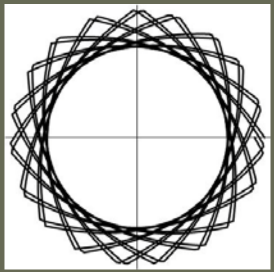
m=e



3 spins



7 spins



14 spins

Third step to superformula

Power Parameters p_i

$$\rho = \left(\left| \frac{1}{a} \cos\left(\frac{m}{4} \varphi \right) \right|^{p_2} + \left| \frac{1}{b} \sin\left(\frac{m}{4} \varphi \right) \right|^{p_3} \right)^{-\frac{1}{p_1}}$$

The number of possible shapes increase greatly assuming different values for the exponents

Each parameter produces the effect of a **non-linear** transformation.

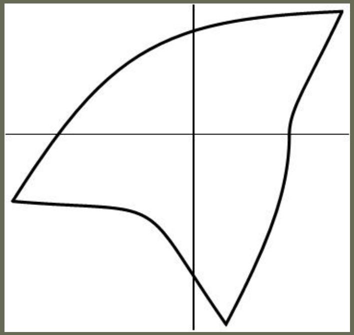
Third step to superformula

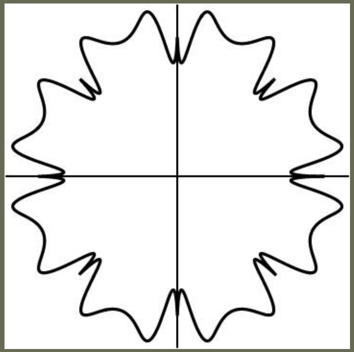
Power Parameters p_i

$$\rho = \left(\left| \frac{1}{a} \cos\left(\frac{m}{4} \varphi \right) \right|^{p_2} + \left| \frac{1}{b} \sin\left(\frac{m}{4} \varphi \right) \right|^{p_3} \right)^{-\frac{1}{p_1}}$$

$m = 3 \; p_1 = 10000$

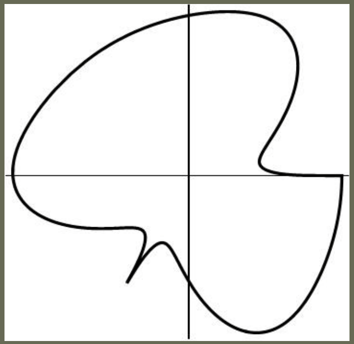
$p_2 = p_3 = 2018$





$m = 16 \quad p_1 = 1$

$p_2 = 10 \; p_3 = 0.3$



$m = 7 \; p_1 = 0.5$

$p_2 = 0.5 \; p_3 = 0.3$

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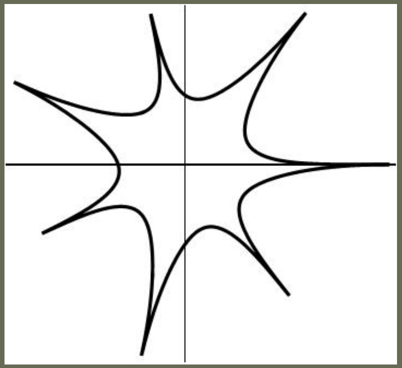
$p_2 = 0.5 \; p_3 = 0.3$



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The superformula

$$\rho = R(\varphi)\left(\left|\frac{1}{a}\cos\left(\frac{m}{4}\varphi\right)\right|^{p_2} + \left|\frac{1}{b}\sin\left(\frac{m}{4}\varphi\right)\right|^{p_3}\right)^{-\frac{1}{p_1}}$$

Two remarkable particular cases:

$$R(\varphi) = \varphi^k \qquad\qquad R(\varphi) = \left|\cos\frac{m}{2}\varphi\right|$$

The superformula

$$\rho = R(\varphi)\left(\left|\frac{1}{a}\cos\left(\frac{m}{4}\varphi\right)\right|^{p_2} + \left|\frac{1}{b}\sin\left(\frac{m}{4}\varphi\right)\right|^{p_3}\right)^{-\frac{1}{p_1}}$$

Two remarkable particular cases:

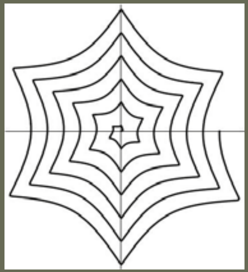
$$R(\varphi) = \varphi$$

$$R(\varphi) = \left|\cos\frac{m}{2}\varphi\right|$$

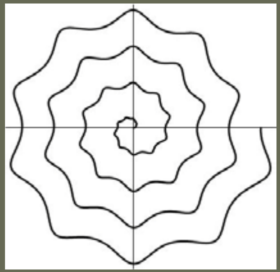
The superformula

$$\rho = \varphi\left(\left|\frac{1}{a}\cos\left(\frac{m}{4}\varphi\right)\right|^{p_2} + \left|\frac{1}{b}\sin\left(\frac{m}{4}\varphi\right)\right|^{p_3}\right)^{-\frac{1}{p_1}}$$

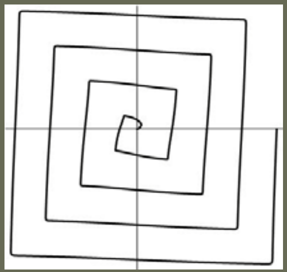
First case: spirals



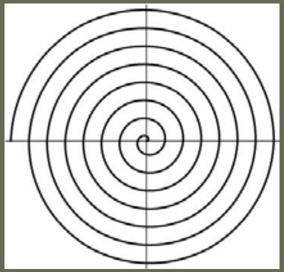
$m = 6\ p_1 = p_2 = p_3 = 100$
$0 \le \varphi \le 12\pi$



$m = 10 \quad p_2 = p_3 = 5$
$p_1 = 8 \quad 0 \le \varphi \le 8\pi$



$m = 4\ p_1 = p_2 = p_3 = 100$
$0 \le \varphi \le 8\pi$



$m = 6 \quad p_2 = p_3 = 1$
$p_1 = 100 \quad 0 \le \varphi \le 15\pi$

$m = 10 \quad p_2 = 50\ p_3 = 5$
$p_1 = 8 \quad 0 \le \varphi \le 16\pi$

$m = 10\ \ p_2 = 0{,}5\ \ p_3 = 2$
$p_1 = 1 \quad 0 \le \varphi \le 16\pi$

The superformula

$$\rho = R(\varphi)\left(\left|\frac{1}{a}\cos\left(\frac{m}{4}\varphi\right)\right|^{p_2} + \left|\frac{1}{b}\sin\left(\frac{m}{4}\varphi\right)\right|^{p_3}\right)^{-\frac{1}{p_1}}$$

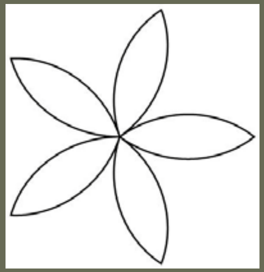
Two remarkable particular cases:

$$R(\varphi) = \varphi \qquad\qquad R(\varphi) = \left|\cos\frac{m}{2}\varphi\right|$$

The superformula

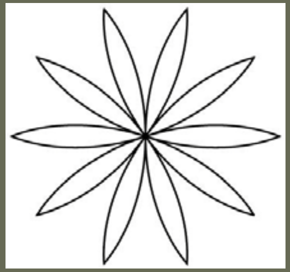
$$\rho = \left|\cos\frac{m}{2}\varphi\right| \left(\left|\frac{1}{a}\cos\left(\frac{m}{4}\varphi\right)\right|^{p_2} + \left|\frac{1}{b}\sin\left(\frac{m}{4}\varphi\right)\right|^{p_3} \right)^{-\frac{1}{p_1}}$$

Second case: flowers



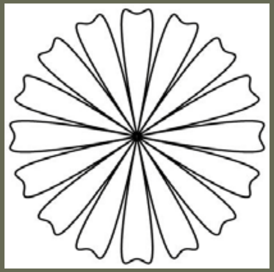
$m = 5$

$p_1 = p_2 = p_3 = 1$



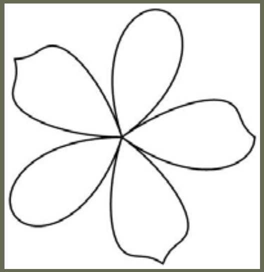
$m = 10$

$p_1 = p_2 = p_3 = 1$



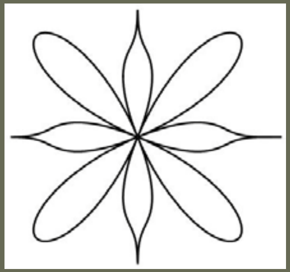
$m = 16$

$p_1 = 1 \;\; p_2 = p_3 = 5$



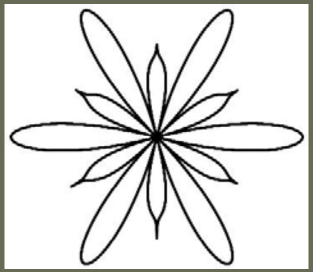
$m = 5$

$p_3 = p_1 = 1 \quad p_2 = 10$



$m = 8$

$p_2 = 5 \;\; p_3 = 0.3 \;\; p_1 = 1$

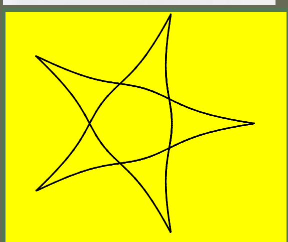


$m = 12$

$p_2 = 0.1 \;\; p_3 = 5 \;\; p_1 = 1$

The superformula

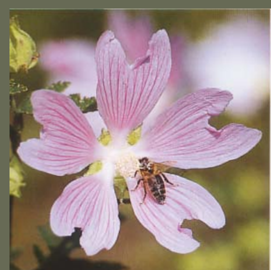
$$\rho = R(\varphi)\left(\left|\frac{1}{a}\cos\left(\frac{m}{4}\varphi\right)\right|^{p_2} + \left|\frac{1}{b}\sin\left(\frac{m}{4}\varphi\right)\right|^{p_3}\right)^{-\frac{1}{p_1}}$$

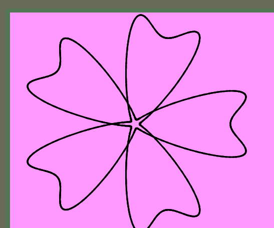


$a = b = 10 \quad m = 5$

$p_1 = p_3 = 2 \quad p_2 = 1$

$0 \le \varphi \le 2 \cdot 2\pi$

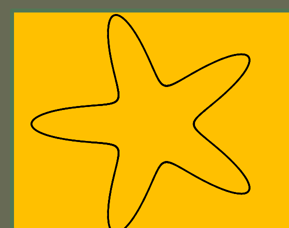




$a = b = 10 \quad m = 5$

$p_1 = 1 \; p_3 = 5.9 \; p_2 = 4.6$

$0 \le \varphi \le 2 \cdot 2\pi$



$a = b = 10 \quad m = 5$

$p_2 = p_3 = 5 \quad p_1 = 1$

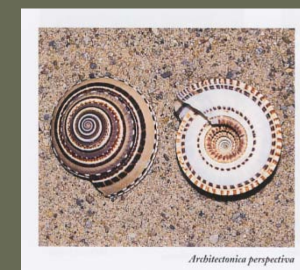
$0 \le \varphi \le 2 \cdot 2\pi$

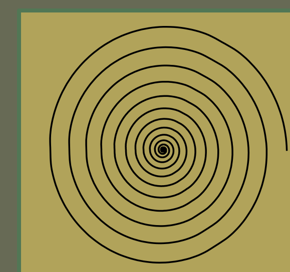
$a = b = 1 \quad m = 10$

$p_2 = p_3 = 5 \quad p_1 = 8$

$0 \le \varphi \le 14 \cdot 2\pi$

$R(\varphi) = \varphi^{2.55}$





$a = b = 1 \quad m = 6$

$p_2 = 0 \quad p_3 = p_1 = 100$

$0 \le \varphi \le 14 \cdot 2\pi$

$R(\varphi) = \varphi^{2.4}$

The superformula

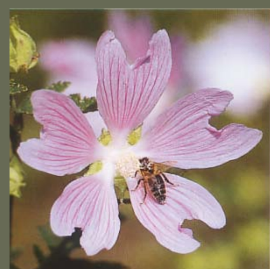
$$\rho = R(\varphi)\left(\left|\frac{1}{a}\cos\left(\frac{m}{4}\varphi\right)\right|^{p_2} + \left|\frac{1}{b}\sin\left(\frac{m}{4}\varphi\right)\right|^{p_3}\right)^{-\frac{1}{p_1}}$$

Calyx and sepals of rose.

$a = b = 10 \quad m = 5$

$p_1 = p_3 = 2 \quad p_2 = 1$

$0 \le \varphi \le 2 \cdot 2\pi$



$a = b = 10 \quad m = 5$

$p_1 = 1 \; p_3 = 5.9 \; p_2 = 4.6$

$0 \le \varphi \le 2 \cdot 2\pi$

$a = b = 10 \quad m = 5$

$p_2 = p_3 = 5 \quad p_1 = 1$

$0 \le \varphi \le 2 \cdot 2\pi$

Pleuroploca trapezium

$a = b = 1 \quad m = 10$

$p_2 = p_3 = 5 \quad p_1 = 8$

$0 \le \varphi \le 14 \cdot 2\pi$

$R(\varphi) = \varphi^{2.55}$

Architectonica perspectiva

$a = b = 1 \quad m = 6$

$p_2 = 0 \quad p_3 = p_1 = 100$

$0 \le \varphi \le 14 \cdot 2\pi$

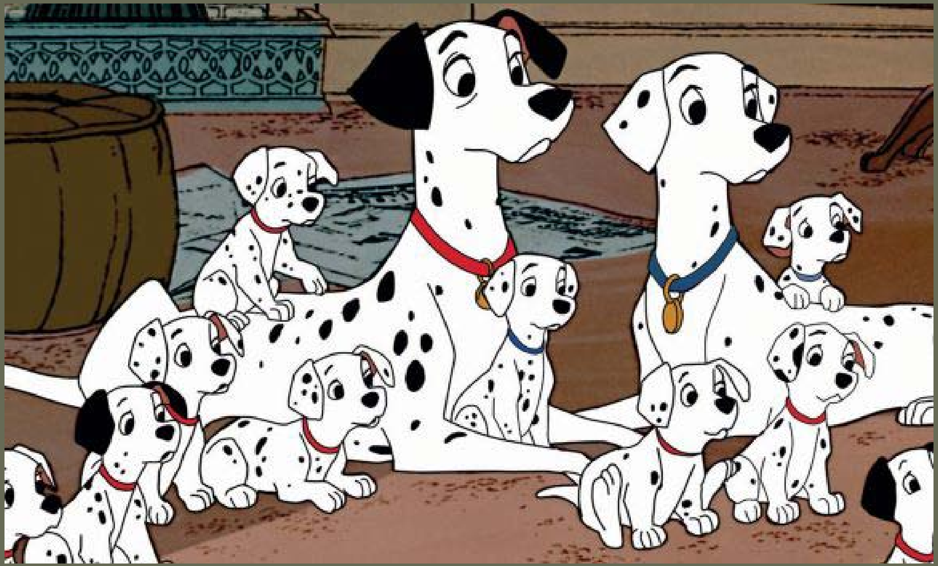
$R(\varphi) = \varphi^{2.4}$

The code

a	b	m	p1	p2	p3	k

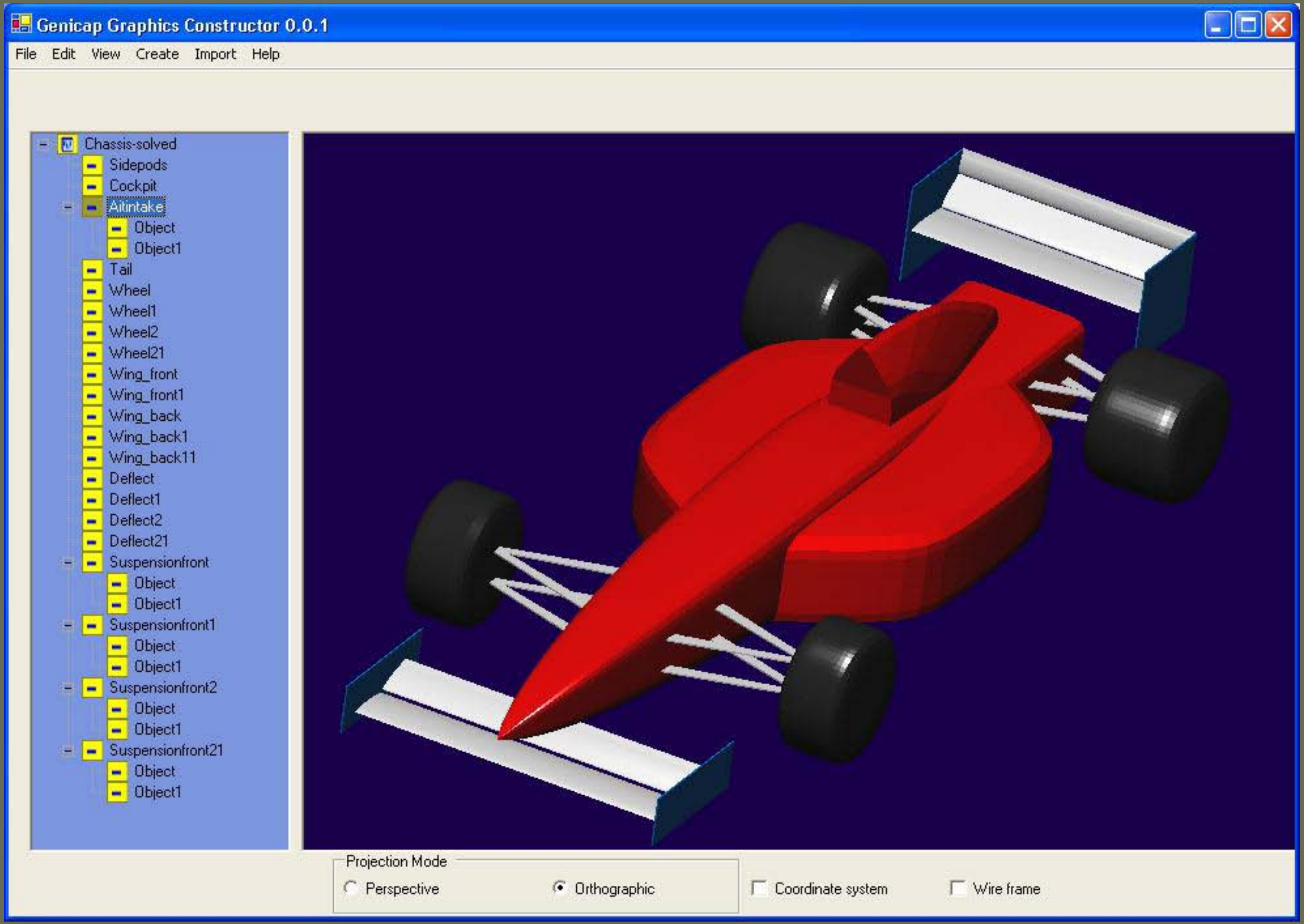
Code – computer graphic

Classis cartoons

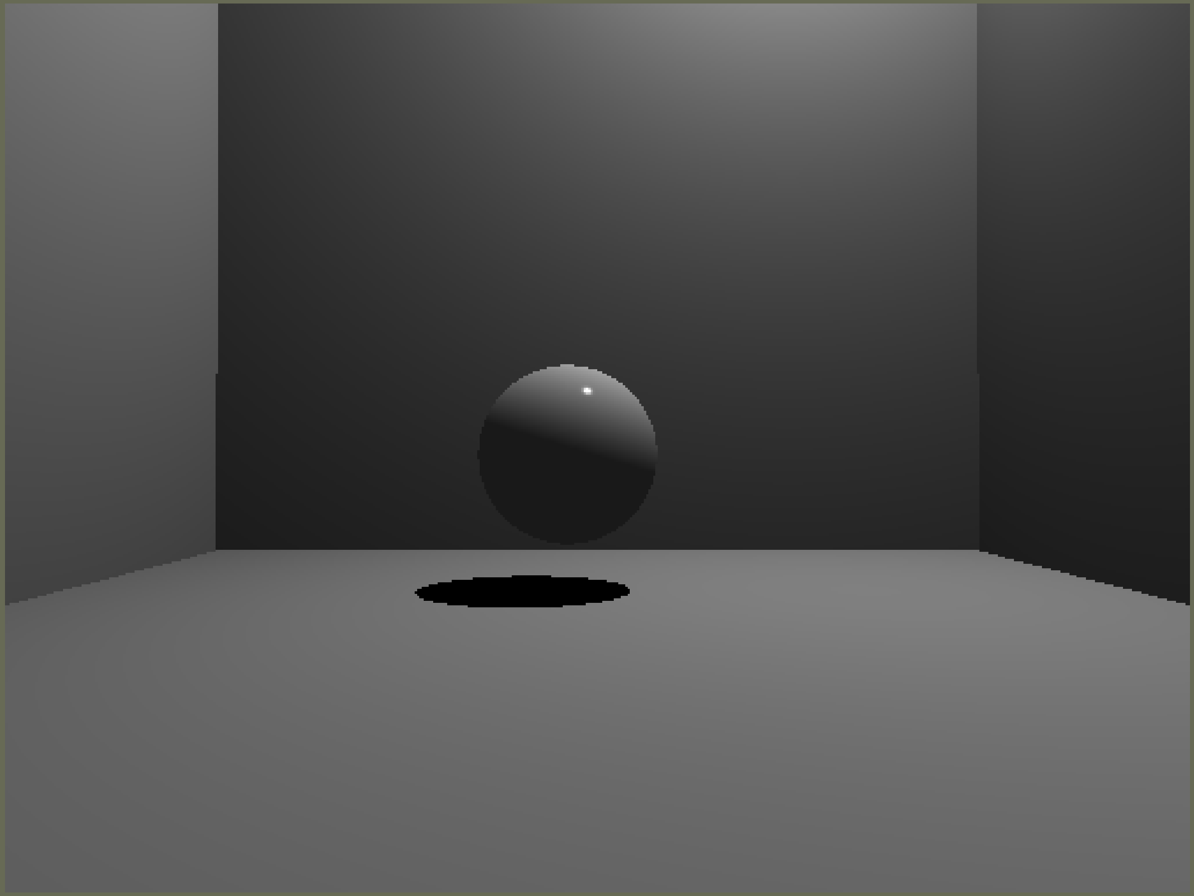


Code – computer graphic

Code – computer graphic



Code – computer graphic



Code – computer graphic

Code – computer graphic

Code – computer graphic

Code – computer graphic

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Thank you very much for your attention

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